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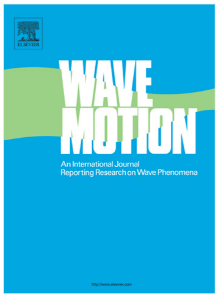

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Jordan - Cattaneo waves: analogues of compressible flow

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Abstract

We review work of Jordan on a hyperbolic variant of the Fisher - KPP equation, where a shock solution is found and the amplitude is calculated exactly. The Jordan procedure is extended to a hyperbolic variant of the Chafee - Infante equation. Extension of Jordan's ideas to a model for traffic flow are also mentioned. We also examine a diffusive susceptible - infected (SI) model, and generalizations of diffusive Lotka - Volterra equations, including a Lotka - Volterra - Bass competition model with diffusion. For all cases we show how a Jordan - Cattaneo wave may be analysed and we indicate how to find the wavespeeds and the amplitudes. Finally we present details of a fully nonlinear analysis of acceleration waves in a Cattaneo - Christov poroacoustic model.

Keywords: acceleration waves, shock waves, Jordan - Cattaneo waves, Chaffee - Infante equation, Lotka - Volterra - Bass competition model, diffusive SI infection

1. Introduction

The concept of a surface of discontinuity, known also as a singular surface, is a very important one in the mechanics of continuous media. The basic idea is an old one as Truesdell and Toupin [1] remark that ... "*surfaces at which derivatives of the velocity are discontinuous first appear in the acoustical researches of Euler*," whereas they also note that ..."*the possibility that the velocity itself may be discontinuous was first remarked by Stokes.*" The basic mechanics including compatibility relations for variables across a singular surface is described in great detail by Truesdell and Toupin [1], pp. 491–529. These writers describe the motion of such a surface of discontinuity and give a detailed account of the kinematics. In particular they, pp. 519–523, describe a singular surface of order one which includes a shock wave where if $x_i = x_i(X_A, t)$ is the motion of the body the velocity $\dot{x}_i$ is discontinuous across the singular surface, $\mathcal{S}$. Truesdell and Toupin [1], pp. 523–525, describe a singular surface of order 2, in which the acceleration $\ddot{x}_i$ is discontinuous across $\mathcal{S}$, and this they term an acceleration wave. The general history of singular surfaces in continuum mechanics is covered in some detail in Truesdell and Toupin [1].

The general theory of acceleration waves in nonlinear elasticity is described in detail by Truesdell and Noll [2], pp. 267–294. Truesdell and Noll [2], p. 267, introduce an acceleration wave in nonlinear elasticity by writing, "*a singular surface of second order with respect to the deformation $x_i = x_i(X_A, t)$ is defined as a surface across which the functions x_i and their first derivatives are continuous, but at least one of the second derivatives $x^i_{,AB}$ suffers a jump discontinuity.*" They recall the kinematic conditions of compatibility for such surfaces, Truesdell and Toupin [1], and note that $[\ddot{x}_i] = U^2 a_i$, "*where the brackets denote the jump occasioned; where the vector* $\mathbf{a}$, *characterizing the strength of the discontinuity, is called the amplitude of the singularity and is assumed different from zero, ..., and where U is called the local speed of propagation. If $U \neq 0$ the surface propagates and is therefore called a wave; ... such a wave must carry a jump of the acceleration, it is called an acceleration wave. It is customary to identify such waves physically with sound waves.*" The history of acceleration waves in nonlinear elasticity is succinctly described by Truesdell and Noll [2]. Of particular note in the theory of acceleration waves is the work of Green [3], who demonstrated that under appropriate conditions within an elastic material the amplitude of an acceleration wave could become infinite in a finite time. This phenomenon is closely connected to the formation of a shock wave, and after the blow-up time the shock would evolve according to the physics of shock waves, cf. Fu and Scott [4], Fu and Scott [5].

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Despite the fact that the mathematics associated to the evolution of an acceleration wave has been known for some time, this is a highly useful technique, largely because it yields *exact* information in a very nonlinear theory in continuum mechanics. Indeed, acceleration waves have been studied recently in many different areas of continuum mechanics or even in biology. For example, they have been proved to be useful in random materials, Nishawala and Ostoja Starzewski [6], Ostoja Starzewski and Trebicki [7, 8]; in saturated porous media, Jordan [9, 10], Jordan et al. [11], Ciarletta and Straughan [12], Ciarletta et al. [13], Straughan and Tibullo [14], Straughan et al. [15]; in hypoplastic materials, Weingartner et al. [16, 17]; in viscoelastic fluids, Gültop et al. [18], Morro [19]; in inhomogeneous fluids, Keiffer et al. [20]; in layers of isotropic solids Currò et al. [21]; in chemotaxis Barbera and Valenti [22]; in plasticity Loret et al. [23]; in perfect gases, relaxing gases and in polytropic gases, Mentrelli et al. [24], Christov et al. [25], Saxena and Jena [26], Shah and Singh [27]; in micro-structured media, Altenbach et al. [28], Eremeyev [29, 30], Eremeyev et al. [31]; in complex materials, Paoletti [32]; in soft materials, Ziv and Shmuel [33]; and in Green-Naghdi fluids, Christov [34], Jordan and Straughan [35].

The outline of the article is that in the next section we review the work of Jordan [36] who showed for the Fisher - KPP equation that one can actually develop a shock wave analysis and the amplitude of the shock can be calculated explicitly. In this sense, the work of Jordan [36] follows previous work on acceleration waves, but now for shock waves. After this we review work on models for traffic flow which employs similar mathematical ideas to those used on the Fisher - KPP equation. This is followed by a shock wave analysis for the Chafee - Infante equation. We then show how the Jordan [36] method may be extended to a susceptible - infected model for the spread of a disease. After this we indicate how similar ideas may be applied to predator - prey models, and in particular, to a model for competition between two products. The article is completed with a fully nonlinear acceleration wave analysis for a model for acoustic wave propagation in a saturated porous material allowing for temperature changes. This work we believe is new and introduces a Darcy type model with the temperature and heat flux being governed by Cattaneo - Christov theory.

2. The Fisher-KPP equation

Fu and Scott [5] write, "*In sharp contrast with acceleration waves, shock waves are very difficult to analyse because their evolutionary behaviour is always coupled with that of the higher order discontinuities that accompany them. If one follows the same procedure as for acceleration waves, one finds that the shock velocity depends on the shock amplitude, whereas the shock amplitude is governed by an evolution equation that also involves the amplitude of the accompanying second order discontinuity.*" A typical system where one encounters the phenomenon described by Fu and Scott [5] is the hyperbolic system for heat propagation where the relaxation time depends on temperature, as analysed by Carillo and Jordan [37] who examine the equations

$$
\begin{aligned}
&\alpha K\theta \frac{\partial q}{\partial t} + q = -K\frac{\partial \theta}{\partial x}\,,\\
&\rho c_p \frac{\partial \theta}{\partial t} + \frac{\partial q}{\partial x} = 0\,,
\end{aligned}
$$

where q, θ, K and c_p are the heat flux, temperature, thermal conductivity, and specific heat at constant pressure. The constant α is a thermal relaxation time, and $\alpha K\theta$ has units of seconds.

In an inspiring piece of work, Jordan [36], showed that a shock wave in a hyperbolic version of the Fisher-KPP equation behaves more like an acceleration wave in that the amplitude satisfies a Bernoulli equation and so may be solved exactly. This work inspired several other articles dealing with other approaches to social problems and biology, such as to gene-culture shock waves, diffusion of information, hunter-gatherer transition, hantavirus evolution, and language diffusion, see Bissell and Straughan [38], Straughan [39, 40, 41], and to a basic SI model for infection in Bargmann and Jordan [42]. We point out that there are several other articles dealing with shock like waves in biological and social situations which treat hyperbolic versions of systems, many of these based on derivations from extended thermodynamics; for example, in predator - prey and reaction - diffusion models, Barbera et al. [43, 44]; in pollution, Barbera et al. [45]; for the hantavirus infection, Barbera et al. [46]; for pattern formation in a semiarid environment, Consolo et al. [47]; in the aquatic food chain, Barbera et al. [48]; in the spread of infectious diseases, Barbera et al. [49]; in chemotaxis, Barbera and Valenti [22]; and in equations for the nerve axon, Zemskov et al. [50].

We now briefly describe the work of Jordan [36] dealing with shock wave evolution. The Fisher - KPP equation deals with the spread of an advantageous gene and is

$$\frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial x^2} = \lambda u\Big(1 - \frac{u}{u_s}\Big), \tag{2.1}$$

where $u(x,t)$ is a density per unit length, and ν, λ and u_s are positive constants, such that $0 \le u \le u_s$. This equation was proposed by Fisher [51] and by Kolmogorov et al. [52]. We employ standard notation throughout and may use $u_t \equiv \partial u/\partial t$, $u_x \equiv \partial u/\partial x$, etc.

Jordan [36] analyses a hyperbolic version of (2.1) by modifying it and writing instead

$$\begin{aligned} &\frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = \lambda u\Big(1 - \frac{u}{u_s}\Big), \\ &\frac{\partial q}{\partial t} = -c_\infty^2 \frac{\partial u}{\partial x}, \end{aligned} \tag{2.2}$$

where q is a flux and c_∞ is a positive constant. Jordan [36] notes that this is equivalent to a Green and Naghdi [53, 54] type II formulation for the flux equation $(2.2)_2$. He also discusses an alternative formulation where $(2.2)_2$ is written instead as

$$\tau \frac{\partial q}{\partial t} + q = -\nu \frac{\partial u}{\partial x}, \tag{2.3}$$

where $\tau > 0$ is a constant relaxation time. This formulation is equivalent to the Cattaneo [55] formulation for the heat flux in continuum mechanics. Care must be taken with the derivation of equation $(2.2)_2$ or (2.3) as is observed critically by Christov [56], Jordan et al. [57], and Su et al. [58]. The reasoning of Cattaneo [55] which is based on statistical mechanics is covered at length in Straughan [41], pp. 11-14. The type II theory for heat evolution in a rigid body is examined at length in Straughan [41], pp. 29-31. Essentially the theory of Green and Naghdi [53] employs in addition to the temperature θ a thermal displacement

$$\alpha = \int_{t_0}^{t} \theta \, ds$$

and a constitutive theory employing θ, α and $\alpha_{,i}$ as independent variables leads effectively to an equation like $(2.2)_2$.

To describe the work of Jordan [36] we define the jump of a function f as

$$\begin{aligned} &= f^- - f^+ , \\ &= \lim_{x \to \mathcal{S}^-} f - \lim_{x \to \mathcal{S}^+} f \end{aligned} \tag{2.4}$$

where $\mathcal{S}$ is the singular surface and $+, -$ refer to the regions ahead and behind the wave. For the system (2.2) the function u is discontinuous across $\mathcal{S}$, the singular surface which defines the shock wave. Jordan [36] begins with the Rankine - Hugoniot equation for system (2.2), see e.g. Whitham [59], p. 31, to find

$$\begin{aligned} &- U[u] + [q] = 0, \\ &- U[q] + c_\infty^2 [u] = 0, \end{aligned} \tag{2.5}$$

where U is the speed of the singular surface. Thus, for non-zero wave amplitude,

$$U = c_\infty . \tag{2.6}$$

If instead one employs the Cattaneo relation (2.3) rather than the Green-Naghdi one $(2.2)_2$, one finds exactly the same wavespeed only now due to notation

$$U = \sqrt{\frac{\nu}{\tau}} . \tag{2.7}$$

Even though the wavespeeds are the same for the Cattaneo or Green-Naghdi formulations, the amplitude equations are not. The Cattaneo formulation contains an extra damping term.

To find the amplitude Jordan [36] takes the jumps of equations (2.2) to find

$$\begin{aligned} &[u_t] + [q_x] = \lambda[u] - \frac{\lambda}{u_s}[u^2], \\ &[q_t] + c_\infty^2[u_x] = 0. \end{aligned} \tag{2.8}$$

Now the Hadamard relation (see Truesdell and Toupin [1]) is used along with the equation for the jump of a product, namely,

$$\frac{\delta}{\delta t}[f] = [f_t] + U[f_x], \tag{2.9}$$

$$[fg] = f^+[g] + g^+[f] + [f][g], \tag{2.10}$$

where $\delta/\delta t$ is the intrinsic derivative and is the rate of change as witnessed by an observer on the wavefront. Define the wave amplitude $A(t)$ by

$$A(t) = [u(t)].$$

Then using the above relations one may derive the amplitude equation

$$\frac{\delta A}{\delta t} = \alpha A - \beta A^2, \tag{2.11}$$

where

$$\alpha = \frac{\lambda}{2} - \frac{u^+}{u_s}\lambda, \qquad \beta = \frac{\lambda}{2u_s},$$

as shown by Jordan [36].

When one employs relation (2.3) then again an equation of form (2.11) is found but now

$$\alpha = \frac{\lambda}{2} - \frac{1}{2\tau} - \frac{u^+}{u_s}\lambda,$$

with β the same as above.

Equation (2.11) is a Bernoulli equation and has exact solution

$$A(t) = \frac{A(0)}{\exp\{-\alpha t\} + (\beta/\alpha)(1 - \exp\{-\alpha t\}A(0))}. \tag{2.12}$$

The various possible behaviours of $A(t)$ are discussed in detail by Jordan [36] who shows that A may blow-up in a finite time. He also shows that the shock amplitude equation exhibits a transcritical bifurcation.

Waves which arise from equations of form $(2.2)_1$ in the general case belong to the class of kinematic waves, see Lighthill and Whitham [60, 61]. Lighthill and Whitham [60] write ... "*we give the theory of a distinctive type of wave motion, which arises in any one - dimensional flow problem where there is an approximate functional relation at each point between the flow quantity q (quantity passing a given point in unit time) and concentration k (quantity per unit distance). The wave property then follows directly from the equation of continuity satisfied by q and k. In view of this, these waves are described as 'kinematic', as distinct from the classical wave motions, which depend also on Newton's second law of motion and are therefore called 'dynamic'.*" The same writers, on p. 282, write, ... "*One important difference is that kinematic viscous waves possess only one wave velocity at each point, while dynamic waves possess at least two (forwards and backwards relative to the medium).*" The waves considered in this article all have forward and backward waves. A very informative example of a kinematic wave is presented in the article of Kaouri [62].

3. Traffic flow

In their classical paper, Lighthill and Whitham [61], *inter alia*, derived a hyperbolic equation for traffic flow on a long one lane road. This equation has form

$$\frac{\partial q}{\partial t} + c\frac{\partial q}{\partial x} + T\frac{\partial^2 q}{\partial t^2} - D\frac{\partial^2 q}{\partial x^2} = 0, \tag{3.1}$$

see Lighthill and Whitham [61], eq. (21). In this equation T is the inertial time for a driver to react before pressing the accelerator or before braking, D is a diffusion coefficient (decrement of flow per unit concentration gradient), c is the speed of the wave, and q is the traffic flow (quantity passing a given point in a unit of time).

Jordan [63] and Christov and Jordan [64] derived an equation not dissimilar to (3.1) by starting with a "density", ρ, of traffic at a point x at time t and a "flux", q, of cars across the point x at time t. They thus have the conservation law

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0\,. \tag{3.2}$$

In addition Jordan [63] and Christov and Jordan [64] adopt a Cattaneo law to govern the flux, of type,

$$\tau \frac{\partial q}{\partial t} + q = v_m \rho \Big(1 - \frac{\rho}{\rho_s}\Big) - \nu \frac{\partial q}{\partial x}\,, \tag{3.3}$$

where τ is the relaxation time, ν is a positive constant, v_m is the maximum speed of a vehicle, and ρ_s is a saturation value for the density of traffic. Note that the saturation term is similar to that in the Fisher - KPP equation. Jordan [63] and Christov and Jordan [64] eliminate q from equations (3.2) and (3.3) to derive the single equation

$$\tau \frac{\partial^2 \rho}{\partial t^2} + \frac{\partial \rho}{\partial t} - \nu \frac{\partial^2 \rho}{\partial x^2} + v_m \rho \Big(1 - \frac{2\rho}{\rho_s}\Big) \frac{\partial q}{\partial x} = 0\,. \tag{3.4}$$

Observe that this has a similarity to the Lighthill and Whitham [61] equation (3.1).

Jordan [63] analyses travelling waves for (3.4), in addition to shock waves and acceleration waves. Christov and Jordan [64] also study shock waves and additionally solve (3.4) numerically using an MUSCL - Hancock procedure. Christov and Jordan [64] write (3.4) as a hyperbolic system

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho \\ q \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} q \\ \nu\rho/\tau \end{pmatrix} = \frac{1}{\tau}\begin{pmatrix} 0 \\ v_m \rho\{1 - \rho/\rho_s\} - q \end{pmatrix} \tag{3.5}$$

Then, if $S(t)$ denotes the shock amplitude, both sets of writers show the shock wavespeed is given by $c_0 = \sqrt{\nu/\tau}$, and they derive a Bernoulli equation for S of form

$$\frac{\delta S}{\delta t} + \mu_0 S + \beta_0 S^2 = 0,$$

where

$$\mu_0 = \frac{1}{2\tau} - \beta_0 + 2\beta_0 \frac{\rho_c^+}{\rho_s}\,, \qquad \beta_0 = \frac{v_m c_0}{2\nu}$$

with $\rho_c^+ \in [0, \rho_s)$ being the value of ρ ahead of the wave evaluated as a limit on the wave.

Tha analysis of Jordan [63] and Christov and Jordan [64] defines the Mach number Ma by $Ma = v_m/c_0$ and they derive detailed shock amplitude behaviour depending on the value of the critical initial amplitude, α^*,

$$\alpha^* = 2\frac{\rho_c^+}{\rho_s} - 1 + \frac{1}{Ma}\,.$$

Jordan [63] concentrated on $\alpha^* > 0$ whereas Christov and Jordan [64] allow all values. Christov and Jordan [64] show S undergoes a transcritical bifurcation and study the shock stability and shock amplitude behaviour carefully.

The numerical results in Christov and Jordan [64] are very detailed. They show that a Taylor shock can form behind the shock wave, and in certain cases a decaying shock may lead to an acceleration wave as $t \to \infty$.

In a not unrelated work Bissell and Straughan [38] analyse the behaviour of a human crowd. In a two - dimensional scenario with $\rho(\mathbf{x}, t)$ the crowd density and $v_i(\mathbf{x}, t)$ the crowd velocity Bissell and Straughan [38] work with the equations

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + v_i \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial v_i}{\partial x_i} = 0, \\ &\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = \frac{1}{\tau}(v_e e_i - v_i) - \frac{c^2}{\rho}\rho_{,i}\,. \end{aligned} \tag{3.6}$$

Here $v_e(\rho)$ and $c(\rho)$ represent a restricting velocity and a crowd pressure term, e_i denote the unit vectors in $\mathbb{R}^2$, and τ is a crowd relaxation time. They derive the acceleration wave speeds in two - dimensions and solve for the wave amplitudes of a one - dimensional wave. They show that under the right circumstances the wave may reach a tipping point where the derivative of the density becomes infinite which amounts to people tipping over each other and effectively witnessing a stampede.

The subject of crowd behaviour is one of much recent study, see e.g. Aylaj et al. [65], Bellomo et al. [66], Bellomo et al. [67], or the collected works edited by Gibelli and Bellomo [68].

4. The Chafee - Infante equation

The partial differential equation

$$\frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial x^2} = \lambda(u - u^3) \tag{4.1}$$

is known as the Chafee - Infante equation, see e.g. Huang and Huang [69], named after original important work on bifucation by Chafee [70] and Chafee and Infante [71, 72].

One may study a hyperbolic generalization of equation (4.1) using either a Green and Naghdi [53, 54] type II formulation for the flux or a Cattaneo [55] one. If we employ a Cattaneo formulation then we write instead of (4.1) the system

$$\begin{aligned} &u_t + J_x = \lambda(u - u^3), \\ &\tau J_t + J + \nu u_x = 0. \end{aligned} \tag{4.2}$$

We develop system (4.2) in the same manner as Jordan [36] to see the Rankine - Hugoniot equations lead to the wavespeed $U = \sqrt{\nu/\tau}$. By then taking the jumps of (4.2) and using the Hadamard and product relations one obtains the amplitude equation

$$\frac{\delta A}{\delta t} = \frac{\lambda}{2}(1 - 3u^{+2})A - \frac{3\lambda u^+}{2}A^2 - \frac{\lambda}{2}A^3\,. \tag{4.3}$$

If we take u^+ constant then we find there are three steady states to equation (4.3). These are given by

$$\bar{A} = 0, \qquad \bar{A} = \frac{1}{2}\Big(-3u^+ \pm \sqrt{4 - 3u^{+2}}\,\Big),$$

where for real $\bar{A}$ we require $u^+ \in [-2/\sqrt{3}, 2/\sqrt{3}]$. If we set $A = \bar{A} + a$ then a linear instability analysis of (4.3) yields the equation

$$\frac{\delta a}{\delta t} = \frac{1}{2}\lambda f a \tag{4.4}$$

where

$$f = -2 + \frac{15}{2}u^{+2} - \frac{3}{2}u^+\sqrt{4 - 3u^{+2}}\,.$$

The graph of f shows $f > 0$ for

$$-2/\sqrt{3} \le u^+ < -(13 - \sqrt{57})/42 \approx -0.36023,$$

and for

$$(13 + \sqrt{57})/42 \approx 0.69949 < u^+ \le 2/\sqrt{3},$$

and so from equation (4.4) we deduce A will grow for u^+ in these intervals. For u^+ in the interval $(-0.36023, 0.69949)$ the function f is negative and we expect linear instability.

In general one will have to solve the nonlinear ordinary differential equation (4.3) numerically. This is typical of what happens when the equation or system of equations becomes more complicated, as for example is found in the hyperbolic co-evolution of a gene and a culture, cf. Straughan [39].

When the solution u is zero ahead of the wave, i.e. $u^+ = 0$, then we may solve (4.3) exactly to find

$$A^2(t) = \frac{e^{\lambda t}(A(0))^2}{1 + (A(0))^2(e^{\lambda t} - 1)}\,.$$

We see that in this case A remains bounded and $A^2 \to 1$ as $t \to \infty$.
Remark 1. One may perform an analogous analysis on a hyperbolic form of the cubic - polynomial equation studied by Rosen [73], namely

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + \lambda(u - u^2)(u - \beta),$$

where D, λ and β are positive constants. In this case one would work with a system of form

$$\begin{aligned} &u_t + J_x = \lambda(u - u^2)(u - \beta),\\ &\tau J_t + J + Du_x = 0. \end{aligned}$$

5. An infection model with diffusion

Mulone et al. [74] considered the nonlinear stability question for an SI model of susceptibles and those with an infection. Let $S(x,t)$ and $I(x,t)$ be the densities of susceptibles and infected, respectively, at position x and at time t. Then the full model considered by Mulone et al. [74], which includes cross diffusion terms, may be written as

$$\begin{aligned} \frac{\partial S}{\partial t} &= a\frac{\partial^2 S}{\partial x^2} + c\frac{\partial^2 I}{\partial x^2} + \mu - \mu S - \beta S I,\\ \frac{\partial I}{\partial t} &= c\frac{\partial^2 S}{\partial x^2} + a\frac{\partial^2 I}{\partial x^2} + \beta S I - (\mu + \epsilon)I. \end{aligned} \tag{5.1}$$

In these equations μ is the recruitment rate of the population and also the per capita death rate, β is the disease transmission coefficient, and ϵ is the disease - induced death rate. The coefficients a and c are positive and a is the diffusion coefficient for S and I in their respective equations, while c is a cross diffusion coefficient representing the weak influence of the gradient of one species upon the other, with $0 < c < a$.

Mulone et al. [74] derived an optimal nonlinear stability result when cross diffusion is absent, $c = 0$. When $c \neq 0$ they constructed a novel Lyapunov functional to demonstrate nonlinear stability. In this work we consider a hyperbolic variant of (5.1) by introducing Cattaneo laws for the flux terms. We concentrate on the case of zero cross diffusion, although the cross diffusion case is considered in remark 1 below.

For a relaxation time $\tau > 0$ and fluxes J and H, related to S and I, respectively, we analyse the system

$$\begin{aligned} &S_t + J_x = \mu - \mu S - \beta S I,\\ &\tau J_t + J + aS_x = 0,\\ &I_t + H_x = \beta S I - (\mu + \epsilon)I,\\ &\tau H_t + H + aI_x = 0. \end{aligned} \tag{5.2}$$

We consider a singular surface $\mathcal{S}$ across which S and I have a finite jump discontinuity. For wavespeed U the Rankine - Hugoniot equations are

$$\begin{aligned} &-U[S] + [J] = 0,\\ &-U\tau[J] + a[S] = 0,\\ &-U[I] + [H] = 0,\\ &-U\tau[H] + a[I] = 0. \end{aligned} \tag{5.3}$$

For non-zero amplitudes we see that $U = \pm\sqrt{a/\tau}$, which corresponds to a shock wave moving to the right and to the left.

To find the amplitudes $[S]$ and $[I]$ we take the jumps of (5.2) and defining

$$A(t) = [S], \qquad B(t) = [I],$$

one may derive the amplitude equations

$$\begin{aligned} 2\frac{\delta A}{\delta t} &= -\Big(\frac{1}{\tau} + \mu + \beta I^+\Big)A - \beta S^+ B - \beta AB,\\ 2\frac{\delta B}{\delta t} &= -\Big(\frac{1}{\tau} + \mu + \epsilon - \beta S^+\Big)B + \beta I^+ A + \beta AB. \end{aligned} \tag{5.4}$$

When $I^+ = 0$, i.e. the wave moves into a region of zero infection with $S^+ = 1$ and we find the steady state solutions of (5.4) are either

$$\bar{A} = 0 \qquad \text{and} \qquad \bar{B} = 0,$$

or

$$\bar{A} = \frac{\tau^{-1} + \mu + \epsilon}{\beta} - S^+ \qquad \text{and} \qquad \bar{B} = \frac{S^+(\tau^{-1} + \mu)}{(\tau^{-1} + \mu + \epsilon)} - \Big(\frac{\tau^{-1} + \mu}{\beta}\Big).$$

In general one will solve (5.4) numerically to determine the evolutionary behaviour of $A(t)$ and $B(t)$.

Remark 2. One may use the Jordan [36] procedure to analyse the full system of equations (5.2). The Rankine - Hugoniot equations lead to *two* waves, each moving to the right and left, with speeds determined from $U^2 = a/\tau \pm c/\tau$. Thus, there is a fast wave and a slow wave. Upon attempting to find the amplitudes, however, one faces the problem pointed out by Fu and Scott [5] (see the beginning of section 2) where the amplitude equations involve not only A and B, but also the jumps of weaker waves, namely $[S_x]$ and $[I_x]$.

Remark 3. For a shock wave in a basic SI model where in (5.2) one sets also $\mu = 0, \epsilon = 0$, and one employs a Green and Naghdi [53, 54] flux, a detailed analysis of amplitude and wavespeed behaviour is given by Bargmann and Jordan [42], who also investigate travelling waves.

Remark 4. The Fisher - KPP equation analysed in section 2 has been employed by Gaudart et al. [75] to model the spread of the Black Death in Europe and in the U.K., see also Coupland [76], section 7.3. Coupland [76] also assesses whether a Jordan - Cattaneo model might be feasible to model the spread of the Black Death.

6. The Lotka-Volterra-Bass competition model

For a density of prey $u(x,t)$ and predator $v(x,t)$ the diffusive Lotka - Volterra system of equations with equal diffusion coefficients may be written as

$$\begin{aligned} u_t &= Du_{xx} + u(a - bv), \\ v_t &= Dv_{xx} + v(hu - c), \end{aligned} \tag{6.1}$$

where D, a, b, c, h are positive constants. These equations are named after the original work of Lotka [77] and Volterra [78] on predator - prey dynamics.

Rothe [79] studied a generalization of this system which includes a logistic growth term (Verhulst [80, 81, 82], Pearl and Reed [83]) for the prey, this system of equations being

$$\begin{aligned} u_t &= Du_{xx} + u(1 - \chi u - v), \\ v_t &= Dv_{xx} + \alpha v(u - 1). \end{aligned} \tag{6.2}$$

A very interesting generalization of these equations is introduced by Dalla Valle [84]. Dalla Valle [84] combines the Lotka - Volterra predator - prey system together with the Bass model for product growth in an economic market, Bass [85], to derive a realistic model for market growth in the pharmacology industry. The model of Dalla Valle [84] is an ordinary differential equation system she calls the LVBC model. If we include equal diffusion coefficients in the LVBC model then one derives the system of equations

$$\begin{aligned} \frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} &= \Big(p_1 + \frac{\alpha_1}{K_1}u\Big)(K_1 - u - \alpha_{12}v)\,, \\ \frac{\partial v}{\partial t} - D\frac{\partial^2 v}{\partial x^2} &= \Big(p_2 + \frac{\alpha_2}{K_2}v\Big)(K_2 - v - \alpha_{21}u)\,. \end{aligned} \tag{6.3}$$

In these equations u and v are cumulative adoptions of products 1 and 2 in a market. The parameters p_1, p_2 identify the innovation of the two products on the market. The terms α_1 and α_2 are growth rates of the products, K_1 and K_2

are the carrying capacities and α_{12} and α_{21} are competition coefficients. Dalla Valle [84] gives estimates for these parameters in a realistic model for pharmaceutical drugs in the health market.

One may use either a Green and Naghdi [53, 54] type II formulation or a Cattaneo [55] formulation to derive a hyperbolic system of equations representing any of the three systems of equations (6.1), (6.2) or (6.3). After that one may employ a Jordan - Cattaneo wave procedure to find the wavespeeds and derive a nonlinear system of ordinary differential equations for the amplitudes, for all three cases. In general, the amplitudes will have to be found numerically.

We outline brief details for the Rothe system (6.2). One introduces fluxes J and H and instead of (6.2) we propose the system, using the Green and Naghdi [53, 54] type II formulation of Jordan [36],

$$\begin{aligned} u_t &= -J_x + u(1-\chi u - v) \\ \tau J_t &= -Du_x \\ v_t &= -H_x + \alpha v(u-1) \\ \tau H_t &= -Dv_x\,. \end{aligned} \tag{6.4}$$

Define the amplitudes A, B, C and E by

$$A(t)=[u], \qquad B(t)=[v], \qquad C(t)=[J], \qquad E(t)=[H].$$

The Rankine - Hugoniot equations yield $UA=C, UB=E$ and the wavespeed satisfies $U=\pm\sqrt{D/\tau}$. We now consider only the wave moving to the right and suppose it is moving into a region of zero predators, i.e. $v^+=0$. Then, after employing the Hadamard and product relations one may show the amplitudes A and B satisfy the system of equations

$$\begin{aligned} 2\frac{\delta A}{\delta t} &= (1-2\chi u^+)A - \chi A^2 - u^+ B - AB \\ 2\frac{\delta B}{\delta t} &= \alpha(u^+-1)B + \alpha AB. \end{aligned} \tag{6.5}$$

The solutions u^+ and v^+ satisfy equations (6.4) and so in general, u^+ can still be the solution to $(6.4)_{1,2}$ with $v^+=0$. However, we consider the constant solution $u^+=1/\chi$. Then there are three steady states $\bar{A}, \bar{B}$ of this system, given by

$$\begin{aligned} &I. \qquad \bar{A}=0, \qquad \bar{B}=0, \\ &II. \qquad \bar{A}=-\frac{1}{\chi}, \qquad \bar{B}=0, \\ &III. \qquad \bar{A}=1-\frac{1}{\chi}, \qquad \bar{B}=1-\chi\,. \end{aligned}$$

In general, the amplitudes may be found numerically from the nonlinear system of ordinary differential equations (6.5).

7. Non - isothermal poroacoustic waves

Acceleration wave analyses for acoustic waves in a non - isothermal saturated porous material are presented in sections 8.2 and 8.3 of Straughan [86]. There he introduces a Jordan - Darcy temperature model, and a poroacoustic model which employs a Cattaneo theory for the heat flux. The work of this section continues in this vein and we again use a Jordan - Darcy model coupled to Cattaneo theory. However, we introduce two new and physically important effects. One, we include viscous dissipation, cf. Nield and Barletta [87]. The second involves the introduction of an invariant derivative for the rate of change of the heat flux. Christov [88] argues that the material derivative for the heat flux in a Cattaneo theory in a moving body should be replaced by a Lie derivative. We do this here and employ what is now known as Cattaneo - Christov theory for the heat flux. Second sound, or the phenomenon whereby heat travels as a wave, is a highly prominent subject at present and is particularly relevant in connection with nanoscale effects, see Sellitto et al. [89].

We employ standard indicial notation in conjunction with the Einstein summation convention throughout this section. Thus, for example, for a velocity field v_i,

$$v_{m,m} \equiv \sum_{m=1}^{3} \frac{\partial v_m}{\partial x_m} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\,,$$

where $\mathbf{v} \equiv (v_1, v_2, v_3) \equiv (u, v, w)$ and $\mathbf{x} \equiv (x_1, x_2, x_3) \equiv (x, y, z)$. A nonlinear example involving the velocity and the density field ρ is

$$v_i\rho_{,i} \equiv \sum_{i=1}^{3} v_i \frac{\partial \rho}{\partial x_i} \equiv u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}\,.$$

Let $\rho(\mathbf{x}, t)$, $v_i(\mathbf{x}, t)$, $T(\mathbf{x}, t)$, and $Q_i(\mathbf{x}, t)$ be the density, velocity, temperature, and heat flux in a saturated porous medium. One may employ equations (2.15), (3.9), (5.3) and (6.1) of Eringen [90] to derive the equations for a Jordan - Darcy material with a Christov [91, 88, 92] derivative for the heat flux. The governing equations may be shown to have form, the balance of mass,

$$\rho_{,t} + v_i\rho_{,i} + \rho v_{i,i} = 0, \tag{7.1}$$

the balance of linear momentum,

$$\rho(v_{i,t} + v_j v_{i,j}) = -p_{,i} - \xi v_i - \gamma T_{,i}\,, \tag{7.2}$$

the equation of energy balance,

$$\rho c_v(T_{,t} + v_i T_{,i}) + Q_{i,i} + p v_{i,i} = -\xi v_i v_i\,, \tag{7.3}$$

and the Cattaneo - Christov equation

$$\tau(Q_{i,t} + v_j Q_{i,j} - Q_j v_{i,j} + Q_i v_{m,m}) + Q_i + kT_{,i} = 0. \tag{7.4}$$

In these equations the pressure is $p = p(\rho, T)$, $\xi = \mu\phi/K$, where μ, ϕ and K are the dynamic viscosity of the saturating fluid, the porosity, and the permeability of the porous medium, c_v is the specific heat at constant volume, τ is the relaxation coefficient, and k is the thermal conductivity of the porous medium. The term in γ arises since the gradient of temperature is a constitutive variable in the theory of Eringen [90].

We are unable to employ the Jordan [36] ideas to develop a shock wave theory for equations (7.1) - (7.4), since the coefficients of the highest derivatives are not constant and depend on the solution itself. This is analogous to the situation of Fu and Scott [5]. However, we are able to progress with a fully nonlinear acceleration wave analysis and solve the problem completely for the wavespeeds and amplitudes, even in the three dimensional case.

We suppose the body occupies a region $\mathcal{B}$ of Euclidean space for all time. Equations (7.1) - (7.4) are assumed to hold on $\mathcal{B} \times (-\infty, \infty)$. For an acceleration wave we suppose that ρ, v_i, T and Q_i are continuous functions of x_i, t on $\mathcal{B} \times (-\infty, \infty)$ and there is a surface $\mathcal{S}(t)$, for each $t \in (-\infty, \infty)$, such that for each $(\mathbf{x}, t)$ a unit normal to $\mathcal{S}$, n_i, is defined at x_i, and the speed of $\mathcal{S}$ at $(\mathbf{x}, t)$ is $u_n(\geq 0)$ in the direction of $\mathbf{n}$. The functions $v_{i,t}, v_{i,j}, Q_{i,t}, Q_{i,j}, \rho_{,t}, \rho_{,i}, T_{,t}$ and $T_{,i}$, and their higher derivatives, are assumed to be continuous functions of x_i, t on $(\mathcal{B} - \mathcal{S}) \times (-\infty, \infty)$ and to have at most jump discontinuities across $\mathcal{S}$. The surface $\mathcal{S}$ is an acceleration wave. The jump of a quantity f is denoted by $[f]$ and is defined as in (2.4).

We make use of singular surface compatibility conditions as are given in the treatise of Truesdell and Toupin [1] and then from Hadamard's lemma and the regularity of ρ, v_i, T and Q_i it may be shown that (details are similar to those in Lindsay and Straughan [93], p. 61),

$$[v_{i,j}] = C_i n_j, \quad [\rho_{,i}] = Bn_i, \quad [T_{,i}] = An_i, \quad [Q_{i,j}] = R_i n_j, \tag{7.5}$$

where the wave amplitudes are defined by

$$A = [n_i T_{,i}], \quad B = [n_i \rho_{,i}], \quad C_i = [n_j v_{i,j}], \quad R_i = [n_j Q_{i,j}]\,. \tag{7.6}$$

To proceed with an acceleration wave analysis we take the jumps of equations (7.1) - (7.4) at $\mathcal{S}$, to find

$$\begin{aligned}
&[\rho_{,t}] + v_i[\rho_{,i}] + \rho[v_{i,i}] = 0,\\
&\rho[v_{i,t}] + \rho v_j[v_{i,j}] = -\frac{\partial p}{\partial \rho}[\rho_{,i}] - \Big(\frac{\partial p}{\partial T} + \gamma\Big)[T_{,i}]\,,\\
&\rho c_v[T_{,t}] + \rho c_v v_i[T_{,i}] + [Q_{i,i}] + p[v_{i,i}] = 0,\\
&\tau[Q_{i,t}] + \tau v_j[Q_{i,j}] - \tau Q_j[v_{i,j}] + \tau Q_i[v_{m,m}] + k[T_{,i}] = 0\,.
\end{aligned} \tag{7.7}$$

For any scalar, vector, or tensor function, the Hadamard relation shows that (see e.g. Lindsay and Straughan [93], p. 62),

$$[\dot\phi] = \frac{\delta}{\delta t}\,[\phi] - (u_n n_k - v_k)[\phi_{,k}]\,, \tag{7.8}$$

where $\delta/\delta t$ is the derivative of a function at the wavefront, i.e.

$$\frac{\delta}{\delta t}\,(\cdot) = \frac{\partial}{\partial t}\,[\cdot] + u_n n_k(\cdot)_{,k}\,,$$

and $\dot\phi = \phi_{,t} + v_i\phi_{,i}$. Now, define U by $U = u_n - v_i n_i$. Then $(7.7)_{1,2}$ yield

$$\begin{aligned}
&-UB + \rho C_i n_i = 0,\\
&-\rho U C_i = -p_\rho B n_i - (p_T + \gamma)A n_i\,.
\end{aligned} \tag{7.9}$$

From $(7.9)_2$ it follows that $C_i = C n_i$, where $C = [v_{a,b}n_a n_b]$. Then (7.9) become

$$\begin{aligned}
&-UB + \rho C = 0,\\
&-\rho U C + p_\rho B + (p_T + \gamma)A = 0\,.
\end{aligned} \tag{7.10}$$

It is now necessary to introduce orthogonal surface coordinates u^α ($\alpha = 1, 2$) on $\mathcal{S}$. Let $x^i_{;\alpha}(= \partial x^i/\partial u^\alpha)$ denote the tangential vectors to $\mathcal{S}$, and then we decompose Q_i and R_i into normal and tangential components as

$$Q_i = Q_\parallel n^i + Q^\alpha_\perp x^i_{;\alpha}, \qquad R_i = R_n n^i + R^\alpha x^i_{;\alpha},$$

where the sum on α over 1 and 2 is understood. From $(7.7)_3$ we now derive

$$-\rho c_v U A + R_n + pC = 0. \tag{7.11}$$

Equation $(7.7)_4$ splits into three equations as follows,

$$-\tau U R_n + kA = 0, \tag{7.12}$$

and

$$-U R_\alpha + Q^\alpha_\perp C = 0, \qquad \alpha = 1, 2. \tag{7.13}$$

Equations (7.10) - (7.12) are a system of four equations in A, B, C and R_n and yield the wavespeeds of $\mathcal{S}$. Equations (7.13) serve to find the tangential components R_α once C is known. From (7.10) - (7.12) the wavespeed U is found to satisfy the equation

$$(U^2 - U_M^2)(U^2 - U_T^2) + \kappa_1 U^2 = 0, \tag{7.14}$$

where $U_M^2 = p_\rho$ and $U_T^2 = k/\rho c_v\tau$ are the squares of the wavespeed of a purely mechanical wave and a purely thermal wave, and

$$\kappa_1 = -\frac{p(p_T + \gamma)}{\rho^2 c_v}\,. \tag{7.15}$$

We expect $p_T + \gamma > 0$ and so $\kappa_1 < 0$. Thus, there are two waves (each moving to the right and left), a fast wave and a slow wave with wavespeeds U_1 and U_2 which satisfy

$$U_2^2 < \{U_T^2, U_M^2\} < U_1^2.$$

One may now progress to calculate the amplitudes themselves. For the full 3-D case the calculations involve much differential geometry, cf. Lindsay and Straughan [93]. To prevent technical details masking the essential physics we restrict attention to a plane wave in one space dimension. The density, velocity, temperature and heat flux are now denoted by ρ, u, T and Q. Let $P = p_\rho/\rho$ and $S = (p_T + \gamma)/\rho$. We differentiate each differential equation with respect to x and take the jump of each result. Define the amplitudes A, B, C and R by

$$A(t) = [T_x], \quad B(t) = [\rho_x], \quad C(t) = [u_x], \quad R(t) = [Q_x].$$

Then, after some calculation and use of the Maxwell relation and product relation for jumps one may derive the following amplitude equations,

$$\frac{\delta B}{\delta t} - u_n[\rho_{xx}] + 2u_x^+ B + 2\rho_x^+ C + 2BC + u[\rho_{xx}] + \rho[u_{xx}] = 0, \tag{7.16}$$

and

$$\begin{aligned}\frac{\delta C}{\delta t} =& - 2P_\rho \rho_x^+ B - P_\rho B^2 - (S_\rho + P_T)(\rho_x^+ A + T_x^+ B + AB)\\ &- 2S_T T_x^+ A - S_T A^2 - \frac{\xi}{\rho} C + \frac{\xi u}{\rho} B,\end{aligned} \tag{7.17}$$

together with

$$\begin{aligned}&c_v \frac{\delta A}{\delta t} - U c_v [T_{xx}] + \frac{1}{\rho}[Q_{xx}] + \frac{p}{\rho}[u_{xx}] + c_v u_x^+ A + c_v T_x^+ C\\ &+ c_v AC - \frac{1}{\rho^2} Q_x^+ B - \frac{1}{\rho^2}\rho_x^+ R - \frac{1}{\rho^2} BR\\ &+ P(\rho_x^+ C + u_x^+ B + BC) + S(T_x^+ C + u_x^+ A + AC)\\ &+ \frac{p}{\rho^2}(\rho_x^+ C + u_x^+ B + BC) + 2\frac{\xi u}{\rho} C - \frac{\xi u^2}{\rho} B = 0,\end{aligned} \tag{7.18}$$

and

$$\tau \frac{\delta R}{\delta t} - \tau U[Q_{xx}] + k[T_{xx}] + R + \tau(u_x^+ R + Q_x^+ C + RC) = 0. \tag{7.19}$$

We now restrict attention to the fast wave moving into a region at rest where ρ =constant, $u = 0$, T =constant, $Q = 0$.

The procedure is now to form the equation $P\times$(7.16)+$U\times$(7.17). Next form the equation $\tau U\times$(7.18)+$\rho^{-1}\times$(7.19). One then multiplies the first equation so obtained by $\tau pU/\rho$ and adds this to $U^2 - \rho P$ times the second equation so obtained.

After some calculation and use of equation (7.14) one may show that the $[\rho_{xx}], [u_{xx}], [T_{,xx}]$ and $[Q_{xx}]$ terms are removed. The next step is to use the equations (7.10) and (7.12) to find B, C and R in terms of A. This then leads to the amplitude equation

$$\frac{\delta A}{\delta t} + \alpha A + \beta A^2 = 0, \tag{7.20}$$

where the constants α and β are given by

$$\alpha = \frac{1}{2(U^4 - U_M^2 U_T^2)}\Big\{\frac{U_T^2(U^2 - U_M^2)}{\tau} - \frac{\mu}{\rho K} U^2(U^2 - U_T^2)\Big\}$$

and

$$\begin{aligned}\beta =& \frac{2p\tau PS^2U^2}{(U^2 - U_M^2)^2} + \frac{1}{(U^2 - U_M^2)}\Big\{\frac{p\tau U^2 S^2}{\rho} - \frac{p\tau U^2 S^2 P_\rho}{\rho}\\ &\qquad - (S_\rho + P_T)p\tau U^2 S + \tau U^2 PS^2\Big\}\\ &+ \tau U^2 S^2 + \tau c_p U^2 S - \frac{p\tau U^2 S_T}{\rho}.\end{aligned}$$

Equation (7.20) is a Bernoulli equation whose solution is well known and presented in many places, e.g. Straughan [41], equation (7.19), or (2.12) of the current article. Under appropriate conditions $A(t)$ may blow up in a finite time, and consequently also $B(t), C(t)$ and $R(t)$, and a shock wave then forms. Details of how an acceleration wave may develop into a shock wave are presented in Fu and Scott [4], Jordan [9], Keiffer et al. [20].

We have thus determined the wavespeeds and derived explicitly the Bernoulli equation (7.20) which leads to an analytical solution for the amplitudes for an acceleration wave for equations (7.1) - (7.4).

8. Conclusions

We have reviewed the work of Jordan [36], and we have shown how his ideas may be extended to other equations and to systems of susceptible - infected populations, and to predator - prey like equations. This may be a useful tool for other systems of partial differential equations where a jump in the actual solution is to be expected. We stress that the results for the amplitude in all cases reported are exact, up to possibly numerical approximation of ordinary differential equations, and do not involve any kind of weakly nonlinear expansion techniques. In addition, a new new model is presented for wave propagation in a non - isothermal saturated porous material. A fully nonlinear acceleration wave analysis is given for this model.

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Work of Jordan on a hyperbolic variant of the Fisher - KPP equation is described, where a shock solution is found and the amplitude is calculated exactly. The Jordan procedure is extended to a hyperbolic variant of the Chafee - Infante equation. Extension of Jordan's ideas to a model for traffic flow is discussed. We also examine a diffusive susceptible - infected (SI) model, and generalizations of diffusive Lotka - Volterra equations, including a Lotka - Volterra - Bass competition model with diffusion. For all cases we show how a Jordan - Cattaneo wave may be analysed and we indicate how to find the wavespeeds and the amplitudes.